Comparative Survey on Fundamental Matrix Estimation[*]

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Abstract

The epipolar geometry is a key point in computer vision and the fundamental matrix estimation the unique way to compute it. This article surveys several methods of fundamental matrix estimation. All the different methods have been programmed and their accuracy analysed using synthetic and real images. A discussion justified with experimental results is given and the code is available in Internet (`http://eia.udg.es/~armangue/research`).

Keywords: Epipolar Geometry, Fundamental Matrix, Performance Evaluation.

1 Introduction

The estimation of three-dimensional information in active systems is a crucial problem in computer vision because the camera parameters might change dynamically depending on the scene. In such situation, only the epipolar geometry, which is contained in the fundamental matrix, can be computed. Basically, the intrinsic parameters of both cameras and the position and orientation of one from the other can be extracted by using Kruppa equations [1]. Moreover, the fundamental matrix can be used to reduce the matching process among the viewpoints [2]. Thus, it is very interesting to develop accurate techniques to compute it. This article surveys fifteen of the most used techniques in computing the fundamental matrix.

The article is organized as follows. First, section two describes all the techniques in a sequence analyzing their advantages and drawbacks with the previous ones. Then, section three deals with the experimental results obtained by using synthetic and real data. The article ends with conclusions.

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2 Estimating the fundamental matrix

In the last few years, several methods to estimate the fundamental matrix have been proposed, which can be classified into lineal methods and iterative methods that deal with bad point localization due to noise in image segmentation, and robust techniques that eliminate the outliers due to false matchings.

Linear methods:

The linear method of the *seven points* is based on computing the fundamental matrix by using only seven point correspondences [3]. Due to the homogeneity of the equations, the solution is a set of matrices of the form $\mathbf{F} = \alpha\mathbf{F}_1 + (1-\alpha)\mathbf{F}_2$. Then, by forcing the rank of the matrix to 2 and using the expression $\det\left[\alpha\mathbf{F}_1 + (1-\alpha)\mathbf{F}_2\right]$ a cubic polynomial is obtained, which has to be solved to obtain α and therefore $\mathbf{F}$. The main advantage of this method is that a fundamental matrix can be estimated by using only seven points, but this fact becomes a drawback when some points are bad located. Moreover, the *7-points method* cannot be applied in the presence of redundancy. Hence, it can not be applied using n points if $n > 7$.

Another interesting method is the *8-points method*, where the redundancy of points permits to minimize the error on estimating $\mathbf{F}$. The equation to minimize is: $\min\limits_{\mathbf{F}} \sum_i \left(m_i^{\mathrm{T}}\mathbf{F}m_i'\right)^2$. The most classical method to solve such equation is the *least-squares technique* by forcing one of the components of $\mathbf{F}$ to be the unity [4]. This simplification can be assumed because $\mathbf{F}$ is always defined up to a scale factor. Then the system to solve is: $f' = (\mathbf{U}'^{\mathrm{T}}\mathbf{U}')^{-1}\mathbf{U}'^{\mathrm{T}}c_9$, where $\mathbf{U}'$ is a matrix containing the first eight columns of $\mathbf{U}$, c_9 is the last column of $\mathbf{U}$ and f' is a vector containing the first eight elements of f. Note that the last element of f is 1. A variant of the *8-points method* can be applied if the equation is solved by using *eigen analysis*, also called *orthogonal least-squares technique* [5]. In this case $\mathbf{F}$ can be determined from the eigen vector corresponding to the smallest eigen value of $\mathbf{U}^{\mathrm{T}}\mathbf{U}$. The difference between this method and the classical *least-squares* resides in the form of calculating the error between correspondences and epipolar lines, where an orthogonal distance to the epipolar line is much more realistic.

The last linear method we surveyed is the *analytic method with rank-2 constraint* [3], which imposes the rank-2 constraint during minimization. The matrix $\mathbf{U}'$ is defined as the composition of the first seven columns of $\mathbf{U}$ and c_8 and c_9 are defined as the eighth and ninth columns of $\mathbf{U}$ respectively, so that $\mathbf{F}$ can be computed as $f' = -f_8(\mathbf{U}'^{\mathrm{T}}\mathbf{U}')^{-1}\mathbf{U}'^{\mathrm{T}}c_8 - f_9(\mathbf{U}'^{\mathrm{T}}\mathbf{U}')^{-1}\mathbf{U}'^{\mathrm{T}}c_9$, where f' is the vector containing the first seven elements of f, and f_8 and f_9 are the eighth and ninth elements of f, respectively. In order to obtain the values of f_8 and f_9, a $\mathbf{F}$ is computed by using the *seven points algorithm*. Then, f is computed by selecting for any choices of pairs of $\mathbf{F}$ the one that minimizes $\|f\| = 1$. This method obtains a rank-2 matrix. However, the *analytic method with rank-2 constraint* does not improve considerably the results of the previously explained methods.

The linear methods are very fast but their accuracy is rather poor in the presence of noise. Better results might be obtain using iterative algorithms.

Iterative methods:

The iterative methods can be classified into two groups. The first group of techniques is based on minimizing the distances between points and epipolar lines, that is $\min_{\mathbf{F}} \sum_i \left(d^2(m_i, \mathbf{F}m'_i) + d^2(m'_i, \mathbf{F}m_i)\right)$.

A first approach consists in applying directly an iterative method as *Newton-Raphson* [6] using *least-squares technique* as initial solution. Another possibility is the *iterative linear method* [3] that is based on computing a weight value w_i equivalent to the epipolar distances by using the previous $\mathbf{F}$ (in the first iteration $w_i = 1$) and then minimize by using *least-squares* in each iteration. Both approaches do not impose the rank-2 constraint. Then, the *nonlinear minimization in parameter space* [3] can solve this situation. This method is based on parameterizing the fundamental matrix keeping in mind that it has rank 2 by fixing just one of the multiple parameterizations. The iteration of this method permits to compute a better rank-2 $\mathbf{F}$. However, it is not enough to obtain a good estimation because the variance of points are not analogous and the least-square technique assume they are comparable. In order to overcome this drawback, the second group of methods have to be considered, which are based on the *gradient-technique* [7]. In this case, the equation to solve is $\min_{\mathbf{F}} \sum_i \left(m_i^{\mathrm{T}}\mathbf{F}m'_i\right)^2 / g_i^2$, where $g_i = ({l_1}^2 + {l_2}^2 + {l'_1}^2 + {l'_2}^2)^{1/2}$. The *gradient-based* technique obtains better results compared to linear methods and the iterative methods of the first group. Although iterative methods are more accurate than linear ones, they are also hard time consuming and they can not eliminate potential outliers. Hence, robust methods have to be considered.

Robust methods:

This paper surveys up to three robust methods: *M-Estimators*, *Least-Median-Squares* (LMedS) and *Random Sampling* (RANSAC), which can be used in the presence of either outliers and bad point localization.

The *M-estimators* [7] tries to reduce the effect of outliers weighting the residual of each point. A lot of different weight functions have been proposed and each one gives a new variant of the M-estimator method. The results obtained are quite good in the presence of outliers, but they are rather bad if the points are bad located. The *LMedS* [3] and *RANSAC* [5] techniques are very similar. First, both techniques are based on selection randomly the set of points that are used to compute $\mathbf{F}$ by using a linear method. The difference between both techniques is in the way of determining the best **F**. The LMedS calculates for each **F** the median of distances between points and epipolar lines and the chosen fundamental matrix has to minimize such a median. The RANSAC calculates for each **F** the number of inliers and the chosen **F** is the one that maximizes it. Once the outliers are eliminated, the **F** is recalculated with the aim of obtaining a better approach. Moreover, another difference is that LMedS is more restrictive than RANSAC, so that it eliminates more points. However, the main constraint of both techniques is their lack of repetitivity due to the aleatory way of selecting the points.

Considerations in F estimation:

Data normalization is a key point in fundamental matrix estimation. It has been proved that the computation should not be applied directly to the raw data in pixels due to potential uncertainties given by huge numbers. Basically, there are two different methods of data normalization. The first method [3] normalize the data between $[-1, 1]$. The second was proposed by Hartley [8] and it is based on two transformations: a) First, the points are translated so that their centroid is placed at the origin; Secondly, the points are scaled so that the mean of the distances of the points to the origin is $\sqrt{2}$. It has been proved that the method proposed by Hartley gives more accurate results than the previous one.

Another interesting fact is that the estimated **F** should be a rank-2 matrix in order to model the epipolar geometry with all the epipolar lines intersecting in the epipole. Although the rank-2 constraint is not imposed in all the surveyed methods, there is a mathematical method that transforms a rank-n square matrix to the closest rank-$(n-1)$ matrix [7]. However, the obtained rank-2 **F** give worse results because it has not been optimized. Then, we propose to use any method which impose a rank-2 matrix in the computation of **F** instead of further transforming it.

3 Experimental Results

The surveyed methods have been programmed and their accuracy analyzed with synthetic and real data, such as underwater images from the seabed obtained by our underwater robot GARBI. Image points have been normalized by using Hartley [8] explained in section two. Table 1 shows the accuracy of each method as the mean and standard deviation of the distances between points and epipolar lines.

The accuracy of the *seven points algorithm* extremely depends on the seven points used. The *least-squares technique* depends inversely on the amount of bad-located points, obtaining usually better results by increasing the amount of points. The *eigen analysis* is the linear method that obtains the best results because an

Table 1: Methods Implemented with mean and std. of error: 1.- seven points; 2.- least-squares (LS) 3.- orthogonal LS; 4.- rank-2 constraint; 5.- iterative lineal using LS; 6.- iterative Newton-Raphson using LS; 7.- minimization in parameter space using eigen; 8.- gradient using LS; 9.- gradient using eigen; 10.- M-Estimator using LS; 11.- M-Estimator using eigen; 12.- M-Estimator proposed by Torr; 13.- LMedS using LS; 14.- LMedS using eigen; 15.- RANSAC using eigen.

Methods	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\sigma = 0.0$ outliers 0%	0.000 0.000	0.000 0.000	0.000 0.000	0.102 0.043	0.000 0.000	0.000 0.000	0.000 0.000	0.000 0.000	0.000 0.000	0.000 0.000	0.000 0.000	0.000 0.000	0.000 0.000	0.000 0.000	0.000 0.000
$\sigma = 0.0$ outliers 10%	22.125 57.007	339.562 433.013	17.124 31.204	30.027 59.471	161.684 117.494	27.035 59.117	17.871 31.225	187.474 197.049	18.224 36.141	73.403 60.443	4.909 4.493	4.714 2.994	0.000 0.000	0.000 0.000	9.449 8.387
$\sigma = 0.1$ outliers 0%	15.048 14.498	1.331 0.788	0.107 0.088	0.120 0.091	1.328 0.786	0.108 0.088	0.112 0.092	1.328 0.786	0.112 0.092	0.355 0.257	0.062 0.042	0.062 0.041	1.331 0.788	0.107 0.088	0.107 0.088
$\sigma = 0.1$ outliers 10%	26.136 66.095	476.841 762.756	19.675 46.505	50.053 53.974	158.671 124.086	70.530 91.194	19.549 46.537	183.961 137.294	15.807 40.301	73.354 59.072	4.876 4.808	4.130 2.997	0.449 0.271	0.098 0.077	9.148 8.564
$\sigma = 0.5$ outliers 0%	15.783 14.837	5.548 3.386	0.538 0.362	0.642 0.528	5.599 3.416	0.538 0.366	0.554 0.361	5.590 3.410	0.554 0.361	2.062 1.466	0.392 0.237	0.367 0.207	5.548 3.386	0.538 0.362	0.538 0.362
$\sigma = 0.5$ outliers 10%	117.534 94.987	507.653 940.808	19.262 49.243	26.475 54.067	161.210 136.828	47.884 65.975	18.933 49.204	217.577 368.061	19.409 51.154	143.442 111.694	3.887 3.969	3.147 2.883	47.418 29.912	0.586 0.434	10.723 12.972
$\sigma = 1.0$ outliers 0%	19.885 16.485	21.275 12.747	1.065 0.744	1.319 0.912	20.757 12.467	1.064 0.747	1.071 0.745	21.234 12.719	1.071 0.745	8.538 6.306	0.794 0.463	0.814 0.463	21.275 12.747	1.065 0.744	1.065 0.744
$\sigma = 1.0$ outliers 10%	138.554 96.671	629.326 833.019	21.264 53.481	61.206 64.583	158.849 120.461	79.323 80.100	20.277 49.476	152.906 120.827	18.730 38.644	120.012 122.436	3.921 3.752	4.089 4.326	25.759 15.217	1.052 0.803	8.657 17.410
Real Image	3.833 4.440	4.683 3.941	1.725 2.138	5.242 4.286	3.068 2.804	2.584 4.768	1.643 2.109	2.949 2.798	1.581 2.056	0.557 0.441	0.650 0.629	0.475 0.368	1.485 1.134	1.039 0.821	1.725 2.138

orthogonal least-squares minimization is more realistic than the classic least-squares. However, all these methods obtain a rank-3 matrix, which means that the epipolar geometry is not properly modeled.

The *analytic method with rank-2 constraint* obtains a rank-2 fundamental matrix. However, the distances between points and epipolar lines are worse than in the linear methods. The *iterative linear method* improves considerably the least-squares technique but can not cope with the outliers problem. The *iterative Newton-Raphson algorithm* obtains even better results than the previous method if there is no outliers present. Although the *nonlinear minimization in parameter space* obtains also a rank-2 matrix, but his computational cost is very high. The eighth and ninth methods are two versions of the *gradient-based method* using least-squares and orthogonal least-squares, respectively. Both methods obtain better results than their equivalent linear methods. Furthermore, the eigen analysis, once again, obtains better results than the other linear methods. Although some of these methods obtain a rank-2 matrix, they can not cope with outliers.

The last surveyed methods are known as "robust", which means they might detect and remove the outliers and compute the fundamental matrix using only the inliers. Three versions of the *M-estimators* have been programmed using least-squares, eigen analysis and the method proposed by Torr [5], respectively. The three methods use a linear initial guess and they become really dependent on the linear method used to estimate it. The following two methods are two versions of LMedS using again least-squares and eigen analysis, respectively. Although the accuracy of *LMedS* seems worse than the one given by *M-estimators*, *LMedS* removes the outliers much more correctly since the epipolar geometry is better modeled. The *RANSAC* is the last surveyed method, which does not obtain better results than *LMedS* with eigen analysis because the method is too permissive selecting the outliers.

4 Conclusions

This article surveys up to fifteen of the most used methods in fundamental matrix estimation. The different methods have been programmed and their accuracy analyzed with synthetic and real images. Experimental results show that: a) linear methods are quite good if the points are well located in the image and the correspondence problem previously solved; b) iterative methods can cope with some gaussian noise in the localization of points, but they become really inefficient in the presence of outliers; c) robust methods can cope with both discrepancy in the localization of points and false matchings.

The experimental results brings out that the orthogonal least-squares using eigen analysis gives better results than the classic least-squares technique of minimization. Moreover, a rank-2 method is preferred because it models the epipolar geometry with all the epipolar lines intersecting at the epipole. Finally, experimental results show up that the corresponding points have to be normalized and the better results have been obtained by using the method proposed by Hartley [7]. Concluding, the best results were obtained with the *LMedS* method forcing the matrix to be rank-2 once the outliers have been removed.

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